

Multibump nodal solutions for an indefinite nonhomogeneous elliptic problem*

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Abstract

We construct multibump nodal solutions of the elliptic equation

$$-\Delta u = a^+[\lambda u + f(\cdot, u)] - \mu a^-g(\cdot, u)$$

in $H_0^1(\Omega)$, when μ is large, under appropriate assumptions, for f superlinear and subcritical and such that the eigenvalues of the associated linearized operator on $H_0^1(\{x \in \Omega : a(x) > 0\})$ at zero, $u \mapsto u - \lambda(-\Delta)^{-1}(a^+u)$, are positive. The solutions are of least energy in some Nehari-type set defined by imposing suitable conditions on orthogonal components of functions in $H_0^1(\Omega)$.

1 Introduction

We are concerned with multibump solutions of the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = a^+[\lambda u + f(\cdot, u)] - \mu a^-g(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

We state our assumptions. The set Ω is an open and bounded Lipschitz domain in \mathbb{R}^N , $N \geq 1$. The function a belongs to $C(\Omega) \cap L^\infty(\Omega)$ and a^+ denotes $\max\{a, 0\}$, $a^- = a^+ - a$ as usual. The set

$$\Omega^+ := \{x \in \Omega : a(x) > 0\}$$

has, say, three components,

$$\Omega^+ = \tilde{\omega} \cup \hat{\omega} \cup \bar{\omega},$$

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also Lipschitz, and

$$\Omega^- := \{x \in \Omega : a(x) < 0\} = \Omega \setminus \overline{\Omega^+}.$$

The value μ is a nonnegative parameter. Let p be a superquadratic and subcritical exponent, $2 < p < 2^*$, where $2^* = +\infty$ if $N = 1$ or $N = 2$, and $2^* = 2N/(N-2)$ if $N \geq 3$. The functions f and g , defined on $\Omega \times \mathbb{R}$, satisfy $f \in C(\Omega^+ \times \mathbb{R})$, $g \in C(\Omega^- \times \mathbb{R})$, f is differentiable with respect to the second variable u in $\Omega^+ \times \mathbb{R}$ and $f' := \partial f / \partial u \in C(\Omega^+ \times \mathbb{R})$. Furthermore, denoting by $F(\cdot, u) := \int_0^u f(\cdot, s) ds$, and $G(\cdot, u) := \int_0^u g(\cdot, s) ds$, the functions f and g satisfy the following hypotheses:

- (a) $\exists_{C_0 > 0} \forall_u |f'(\cdot, u)| \leq C_0(1 + |u|^{p-2})$,
 $\exists_{C'_0 > 0} \forall_u |g(\cdot, u)| \leq C'_0(1 + |u|^{p-1})$.
- (b) $\exists_{\theta > 2} \forall_{u \neq 0} 0 < \theta F(\cdot, u) \leq u f(\cdot, u)$,
 $\exists_{\vartheta > 1} \forall_{u \neq 0} 0 < \vartheta G(\cdot, u) \leq u g(\cdot, u)$.
- (c) $\forall_{u \neq 0} \forall_{x \in \Omega^+} \frac{f(x, u)}{u} < f'(x, u)$.
- (d) Let $\tilde{\lambda}_1, \hat{\lambda}_1, \bar{\lambda}_1$ be the first eigenvalue of $-\Delta u = \lambda a^+ u$ on $H_0^1(\tilde{\omega})$, $H_0^1(\hat{\omega})$, $H_0^1(\bar{\omega})$, respectively. The parameter λ satisfies

$$0 \leq \lambda < \Lambda_1 := \min \left\{ \tilde{\lambda}_1, \hat{\lambda}_1, \bar{\lambda}_1 \right\}.$$

From (b) it follows that $f(\cdot, 0) \equiv f'(\cdot, 0) \equiv g(\cdot, 0) \equiv 0$. Hypothesis (d) is equivalent to saying the parameter λ is nonnegative and smaller than the maximum eigenvalue of the map from $H_0^1(\Omega^+)$ to $H_0^1(\Omega^+)$ defined by $u \mapsto (-\Delta)^{-1}(a^+ u)$; here $(-\Delta)^{-1}$ denotes the inverse of the Dirichlet Laplacian on $H_0^1(\Omega^+)$. An example of functions f and g satisfying our assumptions are

$$\begin{aligned} f(\cdot, u) &= a_1(\cdot)|u|^{p_1-2}u + a_2(\cdot)|u|^{p_2-2}u, \\ g(\cdot, u) &= b_1(\cdot)|u|^{q_1-2}u + b_2(\cdot)|u|^{q_2-2}u, \end{aligned}$$

with $2 < p_1, p_2 < 2^*$, $1 < q_1, q_2 < 2^*$, $a_1, a_2 \in C(\Omega^+) \cap L^\infty(\Omega^+)$, $b_1, b_2 \in C(\Omega^-) \cap L^\infty(\Omega^-)$. In fact, p_1, p_2, q_1, q_2 might even be continuous functions of the space variable, with p_1, p_2 bounded away from 2 and 2^* , and q_1, q_2 bounded away from 1 and 2^* . Our results would still hold if we were to impose less on the function g , namely that it satisfied the inequality in (a) and $G(\cdot, u) \geq c|u|^\vartheta$ for some $c > 0$ and $\vartheta > 1$.

We consider the usual inner product $\langle u, v \rangle = \int \nabla u \cdot \nabla v$ in $H_0^1(\Omega)$, and denote by $\| \cdot \|$ the induced norm. The differential equation in (1.1) is the Euler-Lagrange equation for the energy functional $I_\mu : H_0^1(\Omega) \rightarrow \mathbb{R}$,

$$I_\mu(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \int_\Omega a^+ u^2 - \int_\Omega [a^+ F(\cdot, u) - \mu a^- G(\cdot, u)]. \quad (1.2)$$

Our main result is

Theorem 1.1. *There exists $\check{\mu}$ such that for $\mu > \check{\mu}$ the equation*

$$-\Delta u = a^+[\lambda u + f(\cdot, u)] - \mu a^- g(\cdot, u) \quad (1.3)$$

has an $H_0^1(\Omega)$ weak solution u_μ and, when $\mu_n \rightarrow +\infty$, modulo a subsequence,

$$u_{\mu_n} \rightarrow u \quad \text{in } H_0^1(\Omega), \quad (1.4)$$

where $u|_{\tilde{\omega}}$ is a least energy nodal solution of (1.3) in $H_0^1(\tilde{\omega})$, $u|_{\hat{\omega}}$ is a least energy positive solution of (1.3) in $H_0^1(\hat{\omega})$, and $u|_{\tilde{\omega}}$ and $u|_{\Omega^-}$ are zero.

We assume for simplicity that the set Ω^+ has three components. But when Ω^+ has a different number of components, Theorem 1.1 can be generalized in a way parallel to the one in [13]. In simple terms we may say that, when μ is large, one can choose the solution to be positive, negative, nodal or vanish in any given component of Ω^+ .

Theorem 1.1 generalizes Proposition 2.1 of [13], which addresses the case where $\lambda = 0$, f is homogeneous and $g = f$, more precisely, $f(\cdot, u) = g(\cdot, u) = |u|^{p-2}u$ and $2 < p < 2^*$. Even in this special case we improve our previous results. Also, here all proofs are direct, no argument is by contradiction, so that keeping track of the constants it is possible to give an upper bound for $\check{\mu}$.

We allow for a rather general situation for the nonlinearity. Indefinite weights have also been considered in several other works. The paper [4] concerns existence and multiplicity of positive solutions for elliptic equations whose nonlinear term has the form $W(x)f(u)$ where W changes sign. The paper [3] studies equations with an indefinite nonlinearity both using min-max methods and using Morse theory. In particular, [3] and [4] treat the delicate issue of conditions on the indefinite weight and the nonlinearity that lead to the Palais-Smale condition.

The main ideas for the proof of Theorem 1.1 are from [7], [8], [13] and [17]. More specifically, existence of a sign-changing solution for a superlinear problem was proved in [8] by minimizing the Euler-Lagrange functional over a Nehari-type set. The nonlinearity considered in [8] satisfied conditions similar to the ones imposed to our function f . Using quite a different approach

to ours but an orthogonal decomposition of $H_0^1(\Omega)$, [7] was the first work to establish the existence of multibump *positive* solutions to (1.1), for $N > 1$, when f and g are equal and are homogeneous superlinear functions. The work [17] used cut-off operators and minimization over a Nehari-type manifold to construct positive multispike solutions for an elliptic system. The method of [17] and the orthogonal decomposition of [7] suggested the variational framework in [13], used to prove the existence of multibump nodal solutions to (1.1) in the special case for f and g mentioned above. The technique from [13] is the one we explore here. We would like to emphasize that our solutions are of least energy in a set \mathcal{N}_μ which is not a manifold (see [6, Lemmas 3.1 and 3.2]). In fact, in the present case not even $\mathcal{N}_\mu \cap H^2(\Omega)$ is a manifold, although it does admit a tangent space at the minimum u_μ .

The earliest successes in gluing mountain pass solutions of nonlinear elliptic equations and Hamiltonian systems came from [9], [10] and [18]. The process was simplified by using an alternative procedure in [14], which allowed the authors to glue minimizers on the Nehari manifold together as genuine solutions.

Related local Nehari manifold approaches have already been used in other problems. In [16] a technique which resembles the one in this paper leads to multibump solutions of a semilinear elliptic Dirichlet problem with an operator in divergence form. The solutions are associated to distinct vanishing components of an asymptotically vanishing coefficient. If the degeneration set consists of k connected components, then existence of at least $2^k - 1$ distinct positive solutions, which concentrate on the degeneration set, is established.

It is also important to mention [11], a motivation of [17]. Gluing through local Nehari manifolds was also used in [12].

Recent interesting related results can be found in [1] and [5].

The organization of this paper is as follows. In Section 2 we define the Nehari-type set \mathcal{N}_μ and give estimates for low energy functions. In Section 3 we prove existence of least energy solutions in \mathcal{N}_μ . We also characterize the strong limit of these solutions as $\mu \rightarrow +\infty$. A couple of more technical proofs are left to the Appendix.

2 A Nehari-type set and estimates for low energy functions

Let ϖ be equal to $\tilde{\omega}$, $\hat{\omega}$ or $\bar{\omega}$. Because we assume ϖ is Lipschitz, if $u \in H_0^1(\Omega)$ and $u \equiv 0$ on the complement of ϖ , then $u|_\varpi$ belongs to $H_0^1(\varpi)$. We define

$$\underline{H}(\varpi) = \{u \in H_0^1(\Omega) : u = 0 \text{ in } \Omega \setminus \varpi\}.$$

For u in $H_0^1(\Omega)$, we denote by \tilde{u} , \hat{u} , \bar{u} and \underline{u} the orthogonal projections of u on the orthogonal spaces $\underline{H}(\tilde{\omega})$, $\underline{H}(\hat{\omega})$, $\underline{H}(\bar{\omega})$ and $[\underline{H}(\tilde{\omega}) \oplus \underline{H}(\hat{\omega}) \oplus \underline{H}(\bar{\omega})]^\perp$. The function \underline{u} is harmonic in Ω^+ .

Clearly, the derivative of the energy functional I_μ in (1.2) is

$$I'_\mu(u)(z) = \langle u, z \rangle - \int [a^+[\lambda u + f(\cdot, u)]z - \mu a^-g(\cdot, u)z],$$

for $u, z \in H_0^1(\Omega)$. The solutions u_μ of (1.3) in Theorem 1.1 will be obtained by minimizing the functional I_μ on a Nehari-type set which we will soon define. First we need some parameters. We set $\Lambda = (\lambda/\Lambda_1 + 1)/2 < 1$ and

$$\gamma = \frac{1 - \Lambda}{4}. \quad (2.1)$$

We denote by c_p be the Sobolev constant

$$c_p|u|_p \leq \|u\|,$$

with $|u|_p = (\int u^p)^{1/p}$ the $L^p(\Omega)$ norm of u . When the region of integration is not explicitly indicated, it is understood that integrals are over Ω . Next we will obtain a lower bound for $I_\mu(u)$ when $u \in H_0^1(\Omega^+)$. Consider the set

$$S := \{x \in \Omega^+ : \text{dist}(x, \mathbb{R}^N \setminus \Omega^+) < 1/n_0\}, \quad (2.2)$$

where n_0 is large enough so that

$$|S| \leq \left(\frac{\gamma c_{2^*}^2}{\sup a^+(\lambda + C_0)} \right)^{2^*/(2^*-2)},$$

with C_0 as in (a). Here and henceforth, when $N = 1$ or 2 it should be understood that instead of 2^* a fixed exponent greater than p should appear. For $u \in H_0^1(\Omega)$,

$$\begin{aligned} \left| \int_S a^+[\lambda u + f(\cdot, u)u] \right| &\leq (\lambda + C_0) \int_S a^+|u|^2 + C_0 \int_S a^+|u|^p \\ &\leq \sup a^+(\lambda + C_0)|u|_{2^*}^2 |S|^{(2^*-2)/2^*} + C_0 \int_S a^+|u|^p \\ &\leq \frac{\sup a^+(\lambda + C_0)}{c_{2^*}^2} \|u\|^2 |S|^{(2^*-2)/2^*} + C_0 \int_S a^+|u|^p \\ &\leq \gamma \|u\|^2 + C_0 \int_S a^+|u|^p. \end{aligned} \quad (2.3)$$

There exists a constant C_1 , which we may assume greater than or equal to C_0 , such that

$$\lambda u^2 + f(x, u)u \leq \frac{1}{2}(\lambda + \Lambda_1)u^2 + C_1|u|^p \quad \text{for } u \in \mathbb{R} \text{ and } x \in \Omega^+ \setminus S. \quad (2.4)$$

For $u \in H_0^1(\Omega^+)$.

$$\begin{aligned} \int_{\Omega^+ \setminus S} a^+[\lambda u + f(\cdot, u)]u &\leq \int_{\Omega^+ \setminus S} a^+ \left[\frac{1}{2}(\lambda + \Lambda_1)u^2 + C_1|u|^p \right] \\ &\leq \Lambda \|u\|^2 + C_1 \int_{\Omega^+ \setminus S} a^+|u|^p. \end{aligned} \quad (2.5)$$

Combining (2.3) and (2.5), we obtain

$$\int_{\Omega^+} a^+[\lambda u + f(\cdot, u)]u \leq \frac{3\Lambda+1}{4}\|u\|^2 + C_1 \int_{\Omega^+} a^+|u|^p \quad (2.6)$$

for $u \in H_0^1(\Omega^+)$. We let

$$\rho = \left(\frac{\gamma c_p^p}{\sup a^+ C_1} \right)^{1/(p-2)}.$$

We use inequality (2.6) and (b) to obtain the lower bounds

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2} \int a^+ u^2 - \frac{1}{\theta} \int a^+ f(\cdot, u)u \\ &\geq \frac{1}{2} \left(\|u\|^2 - \lambda \int a^+ u^2 - \int a^+ f(\cdot, u)u \right) \\ &\geq \frac{3\gamma}{2}\|u\|^2 - \frac{C_1}{2} \int a^+ |u|^p \\ &\geq \frac{3\gamma}{2}\|u\|^2 - \frac{\sup a^+ C_1}{2c_p^p} \|u\|^p \\ &\geq \gamma \|u\|^2, \end{aligned} \quad (2.7)$$

for $u \in H_0^1(\Omega^+)$ such that $\|u\| \leq \rho$.

As in [8], one can prove there exists a function $v \in H_0^1(\Omega)$ such that $v = \tilde{v}^+ - \tilde{v}^- + \hat{v}^+$ with \tilde{v}^+ , \tilde{v}^- and $\hat{v}^+ \not\equiv 0$ and

$$I'_\mu(v)(\tilde{v}^+) = I'_\mu(v)(\tilde{v}^-) = I'_\mu(v)(\hat{v}^+) = 0.$$

Finally, we let R satisfy

$$I_\mu(v) < \left(1 - \frac{2}{\theta}\right) (1 - \Lambda)R^2 \quad \text{and} \quad R > \rho.$$

We are ready to give the definition of the Nehari-type set \mathcal{N}_μ .

Definition 2.1. \mathcal{N}_μ is the set of functions $u = \tilde{u} + \hat{u} + \bar{u} + \underline{u} \in H_0^1(\Omega)$ satisfying

$$\begin{aligned}
(\mathcal{N}_i) \quad & I'_\mu(u)(\tilde{u}^+) = I'_\mu(u)(\tilde{u}^-) = I'_\mu(u)(\hat{u}^+) = 0, \\
(\mathcal{N}_{ii}) \quad & \tilde{u}^+, \tilde{u}^-, \hat{u}^+ \not\equiv 0, \\
(\mathcal{N}_{iii}) \quad & I_\mu(u) \leq I_\mu(v) + 1, \\
(\mathcal{N}_{iv}) \quad & \|\tilde{u} + \hat{u}^+\| \leq R, \\
(\mathcal{N}_v) \quad & \max\{\|\hat{u}^-\|, \|\bar{u}\|\} \leq \rho, \\
(\mathcal{N}_{vi}) \quad & \left(\int a^+ \underline{u}^2\right)^{1/2} \leq \gamma \min \left\{ \left(\int a^+ (\tilde{u}^+)^2\right)^{1/2}, \left(\int a^+ (\tilde{u}^-)^2\right)^{1/2}, \left(\int a^+ (\hat{u}^+)^2\right)^{1/2} \right\}, \\
(\mathcal{N}_{vii}) \quad & \|\underline{u}\| \leq \min\{\|\tilde{u}^+\|, \|\tilde{u}^-\|, \|\hat{u}^+\|\}.
\end{aligned}$$

Note that $\mathcal{N}_\mu \neq \emptyset$ as $v \in \mathcal{N}_\mu$. The conditions (\mathcal{N}_{vi}) and (\mathcal{N}_{vii}) are crucial to prove lower bounds on the norms of some of the components of the functions in \mathcal{N}_μ .

The next lemma will allow us to write the integrals $\int a^+ u^2$ and $\int a^+ F(\cdot, u)$, for large μ , as a sum of integrals in terms of the components of u plus a small error.

Lemma 2.2. *For any positive δ , there exists μ_δ such that, for all $\mu > \mu_\delta$,*

$$u \in \mathcal{N}_\mu \quad \Rightarrow \quad |\underline{u}|_p^p < \delta.$$

The proof of Lemma 2.2 is given in the Appendix. In the remainder of this section we establish a number of lemmas which will be used to prove that, for large μ , the functional I_μ has a minimum on \mathcal{N}_μ and to prove that, for large μ , every minimizer of I_μ on \mathcal{N}_μ is a critical point of I_μ . The next lemma will be used (via Lemma 2.6) in connection with (\mathcal{N}_{ii}) and in connection with (\mathcal{N}_{vii}) :

Lemma 2.3. *There exists a positive constant κ_1 such that for all μ ,*

$$u \in \mathcal{N}_\mu \quad \Rightarrow \quad \min \{ \|\tilde{u}^+\|, \|\tilde{u}^-\|, \|\hat{u}^+\| \} \geq \kappa_1.$$

Proof. Denote by w one of the three functions \tilde{u}^+ , $-\tilde{u}^-$ or \hat{u}^+ ; let ϖ be $\tilde{\omega}$ for the first and second choices for w , and be $\hat{\omega}$ for the third choice for w . Note that on the support of w , we have $u = w + \underline{u}$. Let S be as in (2.2). By (\mathcal{N}_{vii}) and a computation similar to (2.3),

$$\int_S a^+ f(\cdot, u) w \leq 2\gamma \|w\|^2 + 2^{p-2} \sup a^+ C_1 \int_S (|w|^p + |\underline{u}|^{p-1} |w|),$$

whereas by (2.4) and (\mathcal{N}_{vi})

$$\begin{aligned}
\int_{\Omega \setminus S} a^+ f(\cdot, u) w &\leq (\gamma + 1) \frac{1}{2} (\Lambda_1 - \lambda) \int a^+ w^2 \\
&\quad + 2^{p-2} \sup a^+ C_1 \int_{\Omega \setminus S} (|w|^p + |\underline{u}|^{p-1} |w|) \\
&\leq (1 - \Lambda)(\gamma + 1) \|w\|^2 \\
&\quad + 2^{p-2} \sup a^+ C_1 \int_{\Omega \setminus S} (|w|^p + |\underline{u}|^{p-1} |w|).
\end{aligned}$$

Similarly, we have

$$\lambda \int a^+ u w \leq (2\Lambda - 1)(\gamma + 1) \|w\|^2.$$

So, by (\mathcal{N}_i) , (2.1) and (\mathcal{N}_{vii}) , a simple computation leads to

$$\|w\|^2 = \lambda \int a^+ u w + \int a^+ f(\cdot, u) w \leq \frac{\Lambda+3}{4} \|w\|^2 + 2^{p-1} \sup a^+ \frac{C_1}{C_p^p} \|w\|^p.$$

From (\mathcal{N}_{ii}) , $\|w\|$ is bounded below by

$$\kappa_1 = \frac{1}{2} \left(\frac{\gamma C_p^p}{2 \sup a^+ C_1} \right)^{1/(p-2)}.$$

□

The next lemma will be used in connection with (\mathcal{N}_{iv}) . We denote by $o(1)$ a quantity whose absolute value can be made arbitrarily small, uniformly in $u \in \mathcal{N}_\mu$, when μ is large.

Lemma 2.4. *Let $\hat{R} > \overline{R} > 0$. If μ is sufficiently large, then*

$$u \in \mathcal{N}_\mu \wedge I_\mu(u) \leq \left(1 - \frac{2}{\theta}\right) (1 - \Lambda) \overline{R}^2 \Rightarrow \|\tilde{u} + \hat{u}^+\| \leq \hat{R}.$$

Proof. Let $u \in \mathcal{N}_\mu$. We bound from below $I_\mu(u)$ by an expression involving

the norms of the components of u . From (b) and (\mathcal{N}_i) ,

$$\begin{aligned}
I_\mu(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2} \int a^+ u^2 - \frac{1}{\theta} \int a^+ f(\cdot, u)u + \mu \int a^- G(\cdot, u) \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \left(\|\tilde{u} + \hat{u} + \bar{u}\|^2 - \lambda \int a^+ (\tilde{u} + \hat{u} + \bar{u})^2 \right) \\
&\quad - \lambda \left(1 - \frac{1}{\theta}\right) \int a^+ (\tilde{u} + \hat{u} + \bar{u})\underline{u} - \frac{1}{\theta} \int a^+ f(\cdot, u)\underline{u} \\
&\quad + \left(\frac{1}{2}\|\underline{u}\|^2 - \frac{\lambda}{2} \int a^+ \underline{u}^2 + \mu \int a^- G(\cdot, \underline{u})\right) \\
&\geq \left(1 - \frac{2}{\theta}\right) (1 - \Lambda) \|\tilde{u} + \hat{u} + \bar{u}\|^2 + o(1).
\end{aligned}$$

For the last inequality we have used condition (a) and Lemma 2.2. This lower bound for $I_\mu(u)$ implies

$$\|\tilde{u} + \hat{u}^+\| \leq \bar{R} + o(1).$$

So

$$\|\tilde{u} + \hat{u}^+\| \leq \hat{R}$$

for sufficiently large μ . □

The next lemma will be used in connection with (\mathcal{N}_v) :

Lemma 2.5. *Let $0 < \delta < 1/2$. If μ is sufficiently large, then for any $u \in \mathcal{N}_\mu$ with $I_\mu(u) < \inf_{\mathcal{N}_\mu} I_\mu + \delta$,*

$$\|\hat{u}^-\| \leq 2\sqrt{\delta/\gamma} \quad \text{and} \quad \|\bar{u}\| \leq 2\sqrt{\delta/\gamma}.$$

Proof. Notice that

$$u \in \mathcal{N}_\mu \quad \Rightarrow \quad u + \hat{u}^- \text{ and } u - \bar{u} \text{ satisfy } (\mathcal{N}_i).$$

From (a) and Lemma 2.2,

$$\begin{aligned}
\int a^+ f(\cdot, u)\hat{u}^- - \int a^+ f(\cdot, -\hat{u}^-)\hat{u}^- &= \int \left(a^+ \int_0^1 f'(\cdot, s\underline{u} - \hat{u}^-) ds \underline{u}\hat{u}^- \right) \\
&= o(1).
\end{aligned}$$

Let $0 < \delta < 1/2$. A simple consequence of (2.7) is that for $u \in \mathcal{N}_\mu$ with $I_\mu(u) < \inf_{\mathcal{N}_\mu} I_\mu + \delta$, with μ sufficiently large so that $u + \hat{u}^- \in \mathcal{N}_\mu$,

$$\begin{aligned}
\inf_{\mathcal{N}_\mu} I_\mu &\leq I_\mu(u + \hat{u}^-) \\
&= I_\mu(u) - \left(\frac{1}{2} \|\hat{u}^-\|^2 - \frac{\lambda}{2} \int a^+ (\hat{u}^-)^2 - \int a^+ F(\cdot, -\hat{u}^-) \right) + o(1) \\
&\leq I_\mu(u) - \gamma \|\hat{u}^-\|^2 + o(1) \\
&< \inf_{\mathcal{N}_\mu} I_\mu + \delta - \gamma \|\hat{u}^-\|^2 + o(1).
\end{aligned} \tag{2.8}$$

Similarly, if μ is sufficiently large, then $u - \bar{u} \in \mathcal{N}_\mu$ and

$$\inf_{\mathcal{N}_\mu} I_\mu \leq I_\mu(u - \bar{u}) \leq I_\mu(u) - \gamma \|\bar{u}\|^2 + o(1) < \inf_{\mathcal{N}_\mu} I_\mu + \delta - \gamma \|\bar{u}\|^2 + o(1). \tag{2.9}$$

Inequalities (2.8) and (2.9) imply Lemma 2.5. \square

The next lemma will be used in connection with (\mathcal{N}_{vi}) :

Lemma 2.6. *There exists a positive constant κ_2 such that for all μ sufficiently large,*

$$u \in \mathcal{N}_\mu \Rightarrow \min \left\{ \left(\int a^+ (\tilde{u}^+)^2 \right)^{1/2}, \left(\int a^+ (\tilde{u}^-)^2 \right)^{1/2}, \left(\int a^+ (\hat{u}^+)^2 \right)^{1/2} \right\} \geq \kappa_2^{1/2}.$$

Proof. Consider again $u \in \mathcal{N}_\mu$ and w equal to one of the three functions \tilde{u}^+ , $-\tilde{u}^-$ or \hat{u}^+ . Let ς be such that $\frac{1}{p} = \frac{\varsigma}{2} + \frac{1-\varsigma}{2^*}$. From Lemma 2.2 and (2.6),

$$\begin{aligned}
\|w\|^2 &= \lambda \int a^+ u w + \int a^+ f(\cdot, u) w = \lambda \int a^+ w^2 + \int a^+ f(\cdot, w) w + o(1) \\
&\leq \frac{3\Lambda+1}{4} \|w\|^2 + C_1 \left(\int a^+ w^2 \right)^{p\varsigma/2} (\sup a^+ |w|_{2^*}^{2^*})^{p(1-\varsigma)/2^*} + o(1).
\end{aligned}$$

Hence, Lemma 2.6 follows from Lemma 2.3. \square

For u and w as above, consider the function $\check{f} : \mathbb{R}^+ \rightarrow \mathbb{R}$, defined by

$$\check{f}(t; w) = \int a^+ \frac{f(\cdot, tw)}{t} w = \int a^+ \frac{f(\cdot, tw)}{tw} w^2,$$

with the understanding that $f(\cdot, u)/u = 0$ for $u = 0$. Henceforth the letter C denotes a constant which may differ from line to line. We examine some simple properties of \check{f} :

Claim 2.7. *If μ is sufficiently large, then (i) the function $\check{f}(t; w) \rightarrow +\infty$ as $t \rightarrow +\infty$, uniformly in μ and $u \in \mathcal{N}_\mu$, (ii) $\check{f}(t; w) \rightarrow \check{f}_0(w)$ as $t \rightarrow 0$, (iii) the function \check{f} is strictly increasing, (iv) there exists κ_0 , independent of μ and $u \in \mathcal{N}_\mu$, such that $\check{f}(1; w) - \check{f}_0(w) \geq \kappa_0$, and (v) there exists κ'_0 , independent of μ and $u \in \mathcal{N}_\mu$, such that $\check{f}'(t; w) \geq \kappa'_0$ for $t \in [\eta, 1/\eta]$; here $0 < \eta < 1$ is fixed.*

Proof. As in (2.2), consider the set

$$\underline{S} := \{x \in \Omega^+ : \text{dist}(x, \mathbb{R}^N \setminus \Omega^+) < 1/\underline{n}\}, \quad (2.10)$$

where \underline{n} is large enough so that

$$|\underline{S}| \leq \left(\frac{\kappa_2 c_{2^*}^2}{2R^2 \sup a^+} \right)^{2^*/(2^*-2)}.$$

We have

$$\int_{\Omega^+ \setminus \underline{S}} a^+ w^2 \geq \kappa_2 - \int_{\underline{S}} a^+ w^2 \geq \kappa_2 - \sup a^+ \frac{R^2}{c_{2^*}^2} |\underline{S}|^{(2^*-2)/2^*} \geq \frac{\kappa_2}{2}. \quad (2.11)$$

Let $0 < \delta < 1$ be fixed. From (b), there exists a constant c_δ such that

$$\frac{f(x, u)}{u} \geq c_\delta |u|^{\theta-2} - \delta \quad \text{for } u \in \mathbb{R} \text{ and } x \in \Omega^+ \setminus \underline{S}.$$

This gives a lower bound for $\check{f}(t; w)$:

$$\begin{aligned} \check{f}(t; w) &\geq c_\delta t^{\theta-2} \int_{\Omega^+ \setminus \underline{S}} a^+ w^\theta - \delta \int_{\Omega^+ \setminus \underline{S}} a^+ w^2 \\ &\geq c_\delta t^{\theta-2} |a^+|_1^{1-\theta/2} \left(\int_{\Omega^+ \setminus \underline{S}} a^+ w^2 \right)^{\theta/2} - \delta \int_{\Omega^+ \setminus \underline{S}} a^+ w^2 \\ &\geq c_\delta t^{\theta-2} |a^+|_1^{1-\theta/2} \frac{\kappa_2^\theta}{2^\theta} - C. \end{aligned}$$

The function $\check{f}(t; w) \rightarrow +\infty$ as $t \rightarrow +\infty$, uniformly in $u \in \mathcal{N}_\mu$ and μ , with μ large. On the other hand, by (a) and the Dominated Convergence Theorem,

$$\check{f}(t; w) \rightarrow \check{f}_0(w) := \int a^+ f'(\cdot, 0) w^2 \quad \text{as } t \rightarrow 0.$$

The function \check{f} is strictly increasing as (c) implies $u \frac{d}{du} \left(\frac{f(\cdot, u)}{u} \right) > 0$. There exists $\kappa_0 > 0$, independent of μ (large) and $u \in \mathcal{N}_\mu$, such that

$$\check{f}(1; w) - \check{f}_0(w) = \int a^+ f(\cdot, w) w - \int a^+ f'(\cdot, 0) w^2 \geq \kappa_0. \quad (2.12)$$

This is a consequence of

Claim 2.8. *Let $0 < \eta < 1$ be fixed. There exists $\kappa'_0 > 0$, independent of μ (large) and $u \in \mathcal{N}_\mu$, such that*

$$\check{f}'(t; w) = \frac{1}{t} \int a^+ \left(f'(\cdot, tw) - \frac{f(\cdot, tw)}{tw} \right) w^2 \geq \kappa'_0, \quad \text{for } t \in [\eta, 1/\eta]. \quad (2.13)$$

The proof of Claim 2.8 is given in the Appendix. \square

Finally, the next lemma will be used in connection with (\mathcal{N}_{vii}) :

Lemma 2.9. *Let $\delta > 0$. If δ is sufficiently small and μ is sufficiently large, then for any $u \in \mathcal{N}_\mu$ with $I_\mu(u) < \inf_{\mathcal{N}_\mu} I_\mu + \delta$,*

$$\|\underline{u}\| \leq 2\sqrt{\delta/\gamma} \quad \text{and} \quad \mu \int a^- G(\cdot, \underline{u}) \leq 2\delta.$$

Proof. For $u \in \mathcal{N}_\mu$, we do not expect $u - \underline{u}$ to belong to \mathcal{N}_μ because this function might not satisfy (\mathcal{N}_i) . We wish to determine \tilde{r} , \tilde{s} and \hat{t} such that

$$\check{u} = \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \bar{u}$$

satisfies (\mathcal{N}_i) . Since $u \in \mathcal{N}_\mu$,

$$\begin{aligned} \|\tilde{u}^+\|^2 - \lambda \int a^+(\tilde{u}^+)^2 &= \int a^+ f(\cdot, \tilde{u}^+) \tilde{u}^+ + o(1), \\ \|\tilde{u}^-\|^2 - \lambda \int a^+(\tilde{u}^-)^2 &= - \int a^+ f(\cdot, -\tilde{u}^-) \tilde{u}^- + o(1), \\ \|\hat{u}^+\|^2 - \lambda \int a^+(\hat{u}^+)^2 &= \int a^+ f(\cdot, \hat{u}^+) \hat{u}^+ + o(1). \end{aligned}$$

The function \check{u} satisfies (\mathcal{N}_i) if

$$\begin{aligned} \tilde{r} \left(\|\tilde{u}^+\|^2 - \lambda \int a^+(\tilde{u}^+)^2 \right) &= \int a^+ f(\cdot, \tilde{r}\tilde{u}^+) \tilde{u}^+, \\ \tilde{s} \left(\|\tilde{u}^-\|^2 - \lambda \int a^+(\tilde{u}^-)^2 \right) &= - \int a^+ f(\cdot, -\tilde{s}\tilde{u}^-) \tilde{u}^-, \\ \hat{t} \left(\|\hat{u}^+\|^2 - \lambda \int a^+(\hat{u}^+)^2 \right) &= \int a^+ f(\cdot, \hat{t}\hat{u}^+) \hat{u}^+, \end{aligned}$$

or

$$\begin{aligned} \check{f}(\tilde{r}; \tilde{u}^+) &= \check{f}(1; \tilde{u}^+) + o(1) > \check{f}_0(\tilde{u}^+), \\ \check{f}(\tilde{s}; -\tilde{u}^-) &= \check{f}(1; -\tilde{u}^-) + o(1) > \check{f}_0(-\tilde{u}^-), \\ \check{f}(\hat{t}; \hat{u}^+) &= \check{f}(1; \hat{u}^+) + o(1) > \check{f}_0(\hat{u}^+). \end{aligned}$$

The last three inequalities (which hold for large μ) follow from (2.12). The properties of the functions \tilde{f} guarantee that the desired \tilde{r} , \tilde{s} and \hat{t} do exist and are unique. The lower bound (2.13) allows us to conclude

$$\tilde{r} = 1 + o(1), \quad \tilde{s} = 1 + o(1), \quad \hat{t} = 1 + o(1). \quad (2.14)$$

Let $0 < \delta < \min\{1/2, (1 - 2/\theta)(1 - \Lambda)R^2 - I_\mu(v)\}$. Suppose $u \in \mathcal{N}_\mu$ with $I_\mu(u) < \inf_{\mathcal{N}_\mu} I_\mu + \delta$. Choose $0 < \overline{R} < \hat{R} < R$ such that $\inf_{\mathcal{N}_\mu} I_\mu + \delta \leq (1 - 2/\theta)(1 - \Lambda)\overline{R}^2$. If μ is sufficiently large, $\tilde{u} \in \mathcal{N}_\mu$ because of (2.14), Lemmas 2.4 and 2.5, and $I_\mu(\tilde{u}) \leq I_\mu(u) + o(1)$. We obtain

$$\begin{aligned} \inf_{\mathcal{N}_\mu} I_\mu &\leq I_\mu(\tilde{u}) \leq I_\mu(\tilde{u} + \hat{u} + \bar{u}) + o(1) \\ &= I_\mu(u) - \frac{1}{2}\|\underline{u}\|^2 + \frac{\lambda}{2} \int a^+ \underline{u}^2 + \int a^+ F(\cdot, \underline{u}) - \mu \int a^- G(\cdot, \underline{u}) + o(1) \\ &\leq \inf_{\mathcal{N}_\mu} I_\mu + \delta - \gamma\|\underline{u}\|^2 - \mu \int a^- G(\cdot, \underline{u}) + o(1). \end{aligned} \quad (2.15)$$

We have used (2.14). Inequality (2.15) implies Lemma 2.9. \square

3 Existence of least energy solutions

For each $u \in \mathcal{N}_\mu$ we define a 3-dimensional manifold \mathcal{M} with global chart $\varphi : \mathbb{R}_+^3 \rightarrow H_0^1(\Omega)$, given by

$$\varphi(\tilde{r}, \tilde{s}, \hat{t}) = \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \bar{u} + \underline{u}. \quad (3.1)$$

Note $\varphi(1, 1, 1) = u$.

Lemma 3.1. *If μ is sufficiently large, the functional $I_\mu|_{\mathcal{M}}$ has a unique absolute maximum. This maximum is strict and attained at u .*

Proof. To evaluate the functional $I_\mu|_{\mathcal{M}}$, we introduce $h : \mathbb{R}_+^3 \rightarrow \mathbb{R}$,

$$\begin{aligned} h(\tilde{r}, \tilde{s}, \hat{t}) &:= I_\mu \circ \varphi(\tilde{r}, \tilde{s}, \hat{t}) \\ &= \frac{\tilde{r}^2}{2} \left(\|\tilde{u}^+\|^2 - \lambda \int a^+ (\tilde{u}^+)^2 \right) + \frac{\tilde{s}^2}{2} \left(\|\tilde{u}^-\|^2 - \lambda \int a^+ (\tilde{u}^-)^2 \right) \\ &\quad + \frac{\hat{t}^2}{2} \left(\|\hat{u}^+\|^2 - \lambda \int a^+ (\hat{u}^+)^2 \right) - \tilde{r}\lambda \int a^+ \tilde{u}^+ \underline{u} + \tilde{s}\lambda \int a^+ \tilde{u}^- \underline{u} \\ &\quad - \hat{t}\lambda \int a^+ \hat{u}^+ \underline{u} - \int a^+ F(\cdot, \tilde{r}\tilde{u}^+ + \underline{u}) - \int a^+ F(\cdot, \underline{u} - \tilde{s}\tilde{u}^-) \\ &\quad - \int a^+ F(\cdot, \hat{t}\hat{u}^+ + \underline{u}) + C_2, \end{aligned} \quad (3.2)$$

with C_2 a constant. From (\mathcal{N}_i) , $\nabla h(1, 1, 1) = 0$. Let ν designate one of \tilde{r} , \tilde{s} or \hat{t} , and accordingly let w designate \tilde{u}^+ , $-\tilde{u}^-$ or \hat{u}^+ . When $\nu = 1$ and no matter what the values of the other two variables,

$$\begin{aligned} \left. \frac{\partial^2 h}{\partial \nu^2} \right|_{\nu=1} &= \|w\|^2 - \lambda \int a^+ w^2 - \int a^+ f'(\cdot, w + \underline{u}) w^2 \\ &= \|w\|^2 - \lambda \int a^+ w^2 - \int a^+ f'(\cdot, w) w^2 + o(1). \end{aligned} \quad (3.3)$$

Indeed, (3.3) follows from

Claim 3.2. *For any positive δ , there exists μ_δ such that, for all $\mu > \mu_\delta$ and $u \in \mathcal{N}_\mu$,*

$$\left| \int a^+ f'(\cdot, w + \underline{u}) w^2 - \int a^+ f'(\cdot, w) w^2 \right| \leq \delta.$$

We leave the simple proof to the reader. Returning to the computation of the second derivative in (3.3), we now use (\mathcal{N}_i) and Lemma 2.2, and afterwards (2.13) for $t = 1$:

$$\begin{aligned} \left. \frac{\partial^2 h}{\partial \nu^2} \right|_{\nu=1} &= - \int a^+ f'(\cdot, w) w^2 + \int a^+ f(\cdot, w) w + o(1) \\ &\leq -\kappa'_0 + o(1) \\ &\leq -\kappa'_0/2, \end{aligned}$$

for μ sufficiently large. Furthermore, we can find $\underline{\nu} < 1 < \overline{\nu}$, independent of $u \in \mathcal{N}_\mu$ for μ large, such that

$$\nu \in [\underline{\nu}, \overline{\nu}] \quad \Rightarrow \quad \frac{\partial^2 h}{\partial \nu^2} \leq -\frac{\kappa'_0}{4}. \quad (3.4)$$

So the function h has a strict local maximum at $(1, 1, 1)$. The function h differs by an $o(1)$ from $\underline{h} : \mathbb{R}_+^3 \rightarrow \mathbb{R}$,

$$\begin{aligned} \underline{h}(\tilde{r}, \tilde{s}, \hat{t}) &:= \frac{\tilde{r}^2}{2} \left(\|\tilde{u}^+\|^2 - \lambda \int a^+ (\tilde{u}^+)^2 \right) + \frac{\tilde{s}^2}{2} \left(\|\tilde{u}^-\|^2 - \lambda \int a^+ (\tilde{u}^-)^2 \right) \\ &\quad + \frac{\hat{t}^2}{2} \left(\|\hat{u}^+\|^2 - \lambda \int a^+ (\hat{u}^+)^2 \right) + C_2 \\ &\quad - \int a^+ F(\cdot, \tilde{r}\tilde{u}^+) - \int a^+ F(\cdot, -\tilde{s}\tilde{u}^-) - \int a^+ F(\cdot, \hat{t}\hat{u}^+). \end{aligned}$$

For large μ , \underline{h} also must have a strict local maximum in $[\underline{\nu}, \overline{\nu}]^3$, say at $(\tilde{r}_1, \tilde{s}_1, \hat{t}_1)$ (dependent on μ and u , of course). It is simple to check using

(c) that if $\partial \underline{h} / \partial \nu = 0$, then $\partial^2 \underline{h} / \partial \nu^2 < 0$. This implies that $\partial \underline{h} / \partial \nu > 0$ for $\nu < \check{\nu}$ and $\partial \underline{h} / \partial \nu < 0$ for $\nu > \check{\nu}$. Again, we use the fact that h is uniformly close to \underline{h} to see that the maximum of h at $(1, 1, 1)$ is unique and absolute. We have proved Lemma 3.1. \square

Proposition 3.3. *Let (u_n) be a minimizing sequence for I_μ restricted to \mathcal{N}_μ . Then, modulo a subsequence, for sufficiently large μ , $u_n \rightarrow u$ in $H_0^1(\Omega)$ and u is a minimizer.*

Proof. Let (u_n) be a minimizing sequence for I_μ restricted to \mathcal{N}_μ , $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Let w_n be \tilde{u}_n^+ , $-\tilde{u}_n^-$ or \hat{u}_n^+ and, accordingly, let w be \tilde{u}^+ , $-\tilde{u}^-$ or \hat{u}^+ and ν be \tilde{r} , \tilde{s} or \hat{t} . Suppose that

$$\|w\| < \liminf \|w_n\|. \quad (3.5)$$

Lemma 2.6 gives

$$\|w\| \geq \kappa_2^{1/2} \Lambda_1^{1/2}. \quad (3.6)$$

The function u will not satisfy (\mathcal{N}_i) because

$$\|w\|^2 - \lambda \int a^+ w^2 - \lambda \int a^+ \underline{u} w - \int a^+ f(\cdot, w + \underline{u}) w < 0.$$

We define the value

$$\nu_0 = \frac{1}{R} \left(\frac{\gamma c_p^p}{\sup a^+ C_1} \right)^{1/(p-2)} = \frac{\rho}{R}.$$

As in (2.6),

$$\begin{aligned} & \nu_0^2 \left(\|w\|^2 - \lambda \int a^+ w^2 \right) - \nu_0 \left(\lambda \int a^+ \underline{u} w - \int a^+ f(\cdot, \nu_0 w + \underline{u}) w \right) \\ &= \nu_0^2 \left(\|w\|^2 - \lambda \int a^+ w^2 - \int a^+ \frac{f(\cdot, \nu_0 w)}{\nu_0} w \right) \\ & \quad - \nu_0 \left(\lambda \int a^+ \underline{u} w + \int a^+ f(\cdot, \nu_0 w + \underline{u}) w - \int a^+ f(\cdot, \nu_0 w) w \right) \\ & \geq \nu_0^2 \left(\|w\|^2 - \frac{3\Lambda+1}{4} \|w\|^2 - C_1 \nu_0^{p-2} \int a^+ |w|^p \right) + o(1) \\ & \geq 2\gamma \nu_0^2 \kappa_1^2 + o(1) \\ & > 0, \end{aligned}$$

for large μ . By continuity, there will exist $\nu_1 \in]\nu_0, 1[$ such that

$$\nu_1 \left(\|w\|^2 - \lambda \int a^+ w^2 \right) - \lambda \int a^+ \underline{u} w - \int a^+ f(\cdot, \nu_1 w + \underline{u}) w = 0.$$

(The value of ν_1 depends on w , but ν_0 is fixed.) Hence there exist

$$\tilde{r}_1, \tilde{s}_1, \hat{t}_1 \in]\nu_0, 1] \quad (3.7)$$

such that

$$\check{u} := \tilde{r}_1 \tilde{u}^+ - \tilde{s}_1 \tilde{u}^- + \hat{t}_1 \hat{u}^+ - \hat{u}^- + \bar{u} + \underline{u}$$

satisfies (\mathcal{N}_i) . It also satisfies (\mathcal{N}_{ii}) because of (3.6). We estimate the energy of \check{u} using (3.5) and Lemma 3.1 applied to u_n :

$$\begin{aligned} I_\mu(\check{u}) &< \liminf I_\mu(\tilde{r}_1 \tilde{u}_n^+ - \tilde{s}_1 \tilde{u}_n^- + \hat{t}_1 \hat{u}_n^+ - \hat{u}_n^- + \bar{u}_n + \underline{u}_n) \\ &\leq \lim I_\mu(u_n) \\ &= \inf I_\mu|_{\mathcal{N}_\mu}. \end{aligned} \quad (3.8)$$

So \check{u} satisfies (\mathcal{N}_{iii}) . The function \check{u} satisfies (\mathcal{N}_{iv}) because of (3.7), and it clearly satisfies (\mathcal{N}_v) . It satisfies (\mathcal{N}_{vi}) for large μ because of Lemmas 2.2 and 2.6 and of the strong convergence of u_n to u in $L^2(\Omega)$. Applying Lemma 2.9 to u_n for large n , with $\delta = \nu_0^2 \kappa_2 \Lambda_1 / (4\gamma)$ and μ sufficiently large, and using (3.6) and the weak lower semi-continuity of $\|\underline{u}_n\|$, we obtain $\|\underline{u}\| \leq \nu_0 \kappa_2^{1/2} \Lambda_1^{1/2} \leq \nu_0 \min\{\|\tilde{u}^+\|, \|\tilde{u}^-\|, \|\hat{u}^+\|\}$. So \check{u} also satisfies (\mathcal{N}_{vii}) . In conclusion, \check{u} belongs to \mathcal{N}_μ and inequality (3.8) is impossible. Thus, $\|w\| = \liminf \|u_n\|$ and $u \in \mathcal{N}_\mu$ is a minimizer of I_μ restricted to \mathcal{N}_μ . In fact, if $\|u\|$ were to be smaller than $\liminf \|u_n\|$, due to a drop in $\|\hat{u}_n^-\|$, $\|\bar{u}_n\|$ or $\|\underline{u}_n\|$ upon passing to the limit, then we would still have strict inequality in (3.8), and again a contradiction. We have proved Proposition 3.3. \square

Proposition 3.4. *If μ is sufficiently large, every minimizer of I_μ on \mathcal{N}_μ is a critical point of I_μ .*

Proof. Let μ be large enough so that Proposition 3.3 holds. Let u be a minimizer of I_μ restricted to \mathcal{N}_μ . Consider the maps $\tilde{J}_\mu^+, \tilde{J}_\mu^-, \hat{J}_\mu^+ : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tilde{J}_\mu^+(z) = I'_\mu(z) \tilde{z}^+, \quad \tilde{J}_\mu^-(z) = -I'_\mu(z) \tilde{z}^-, \quad \hat{J}_\mu^+(z) = I'_\mu(z) \hat{z}^+,$$

and $J_\mu : [\underline{\nu}, \bar{\nu}]^3 \rightarrow \mathbb{R}^3$ defined by

$$J_\mu(\tilde{r}, \tilde{s}, \hat{t}) = (\tilde{J}_\mu^+, \tilde{J}_\mu^-, \hat{J}_\mu^+) \circ \varphi(\tilde{r}, \tilde{s}, \hat{t}) = \left(\tilde{r} \frac{\partial h}{\partial \tilde{r}}, \tilde{s} \frac{\partial h}{\partial \tilde{s}}, \hat{t} \frac{\partial h}{\partial \hat{t}} \right).$$

Here the maps φ and h are the ones corresponding to u as in (3.1) and (3.2). Using (3.4) and $\nabla h(1, 1, 1) = 0$, we can find ν_2 and ν_3 , independent of μ , with $\underline{\nu} \leq \nu_2 < 1$ and $1 < \nu_3 \leq \bar{\nu}$, such that

$$(\tilde{r}, \tilde{s}, \hat{t}) \in [\nu_2, \nu_3]^3 \quad \Rightarrow \quad \frac{\partial(\tilde{J}_\mu^+ \circ \varphi)}{\partial \tilde{r}}, \frac{\partial(\tilde{J}_\mu^- \circ \varphi)}{\partial \tilde{s}}, \frac{\partial(\hat{J}_\mu^+ \circ \varphi)}{\partial \hat{t}} \leq -\frac{\kappa'_0}{8}. \quad (3.9)$$

It follows that on the boundary of $[\nu_2, \nu_3]^3$ either one of the components of J_μ is greater than $\frac{\kappa'_0}{8}(1 - \nu_2)$, or one of the components of J_μ is less than $-\frac{\kappa'_0}{8}(\nu_3 - 1)$ and

$$\deg(J_\mu, [\nu_2, \nu_3]^3, 0) = -1.$$

Suppose that $I'_\mu(u) \neq 0$. Let $B_{\hat{\rho}}(u) := \{z \in H_0^1(\Omega) : \|z - u\| < \hat{\rho}\}$. Choose $\hat{\rho} > 0$ satisfying $I'_\mu(z) \neq 0$ for all $z \in B_{\hat{\rho}}(u)$,

$$\hat{\rho} < \text{dist}(u, \varphi(\mathbb{R}_+^3 \setminus [\nu_2, \nu_3]^3)), \quad (3.10)$$

and so that conditions $(\mathcal{N}_{ii}) - (\mathcal{N}_{vii})$ hold for all $z \in B_{\hat{\rho}}(u)$. This is possible because $u \in \mathcal{N}_\mu$ ((\mathcal{N}_{ii})), $I_\mu(u) \leq I_\mu(v)$ ((\mathcal{N}_{iii})), of Lemma 2.4 ((\mathcal{N}_{iv})), of Lemma 2.5 ((\mathcal{N}_v)), of Lemmas 2.2 and 2.6 ((\mathcal{N}_{vi})), and of Lemmas 2.3 and 2.9 ((\mathcal{N}_{vii})). Note that the choice of $\hat{\rho}$ might depend on μ . Let $\phi : H_0^1(\Omega) \rightarrow [0, 1]$ be Lipschitz, $\phi = 1$ on $B_{\hat{\rho}/2}(u)$ and $\phi = 0$ on $H_0^1(\Omega) \setminus B_{\hat{\rho}}(u)$ and let $K_\mu : B_{\hat{\rho}}(u) \rightarrow H_0^1(\Omega)$ be a pseudogradient vector field for I'_μ on $B_{\hat{\rho}}(u)$. Consider the Cauchy problem

$$\begin{cases} \frac{d\eta}{d\tau} = -\phi(\eta)K_\mu(\eta), \\ \eta(0) = z, \end{cases}$$

for $z \in H_0^1(\Omega)$; by definition, ϕK_μ is zero outside $B_{\hat{\rho}}(u)$. We denote the solution of this Cauchy problem by $\eta(\tau; z)$. For $\tau > 0$, let

$$\varphi_\tau(\tilde{r}, \tilde{s}, \hat{t}) = \eta(\tau; \varphi(\tilde{r}, \tilde{s}, \hat{t})).$$

Each φ_τ is continuous and, due to (3.10),

$$\varphi_\tau|_{\partial([\nu_2, \nu_3]^3)} = \varphi|_{\partial([\nu_2, \nu_3]^3)}$$

and so

$$\deg(J_\mu^\tau, [\nu_2, \nu_3]^3, 0) = \deg(J_\mu, [\nu_2, \nu_3]^3, 0) = -1,$$

where

$$J_\mu^\tau(\tilde{r}, \tilde{s}, \hat{t}) := (\tilde{J}_\mu^+, \tilde{J}_\mu^-, \hat{J}_\mu^+) \circ \varphi_\tau(\tilde{r}, \tilde{s}, \hat{t}).$$

It follows that there exists some $(\tilde{r}_1, \tilde{s}_1, \hat{t}_1) \in]\nu_2, \nu_3[^3$, with $\varphi_\tau(\tilde{r}_1, \tilde{s}_1, \hat{t}_1)$ satisfying (\mathcal{N}_i) . The function $\varphi_\tau(\tilde{r}_1, \tilde{s}_1, \hat{t}_1)$ has to belong to $B_{\hat{\rho}}(u)$ as outside $B_{\hat{\rho}}(u)$ the maps φ and φ_τ coincide, and φ only satisfies (\mathcal{N}_i) in $[\nu_2, \nu_3]^3$ at the point $(1, 1, 1)$. This is a consequence of (3.9). But on $B_{\hat{\rho}}(u)$ conditions $(\mathcal{N}_{ii}) - (\mathcal{N}_{vii})$ hold, so $\varphi_\tau(\tilde{r}_1, \tilde{s}_1, \hat{t}_1)$ belongs to \mathcal{N}_μ . By Lemma 3.1, the maximum of $I_\mu \circ \varphi$ is strict and attained at $(1, 1, 1)$. For $\tau > 0$, $\max I_\mu \circ \varphi_\tau < I_\mu(u) = \min I_\mu|_{\mathcal{N}_\mu}$. This contradicts $\varphi_\tau(\tilde{r}_1, \tilde{s}_1, \hat{t}_1) \in \mathcal{N}_\mu$. We have proved Proposition 3.4. \square

Proof of Theorem 1.1. By Propositions 3.3 and 3.4, there exists $\check{\mu}$ such that for $\mu > \check{\mu}$ the equation (1.3) has an $H_0^1(\Omega)$ weak solution u_μ . Suppose $\mu_n \rightarrow +\infty$ and u_{μ_n} is a minimizer of I_{μ_n} restricted to \mathcal{N}_{μ_n} . Modulo a subsequence,

$$u_{\mu_n} \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

It is clear from Lemmas 2.5 and 2.9 that

$$u = \tilde{u} + \hat{u}^+,$$

and

$$\begin{aligned} I_{\mu_n}(u_{\mu_n}) &= \frac{1}{2} \|\tilde{u}_{\mu_n}\|^2 - \frac{\lambda}{2} \int a^+(\tilde{u}_{\mu_n})^2 + \frac{1}{2} \|\hat{u}_{\mu_n}^+\|^2 - \frac{\lambda}{2} \int a^+(\hat{u}_{\mu_n}^+)^2 \\ &\quad - \int a^+ F(\cdot, \tilde{u}_{\mu_n}) - \int a^+ F(\cdot, \hat{u}_{\mu_n}^+) + o(1). \end{aligned} \quad (3.11)$$

Obviously from Lemma 2.6

$$\tilde{u}^+, \tilde{u}^-, \hat{u}^+ \not\equiv 0.$$

Suppose that either one of the two inequalities

$$\|\tilde{u}\| < \liminf \|\tilde{u}_{\mu_n}\| \quad \text{or} \quad \|\hat{u}^+\| < \liminf \|\hat{u}_{\mu_n}^+\| \quad (3.12)$$

is satisfied. Then

$$\begin{aligned} I_0(u) &= \frac{1}{2} \|\tilde{u}\|^2 - \frac{\lambda}{2} \int a^+ \tilde{u}^2 + \frac{1}{2} \|\hat{u}^+\|^2 - \frac{\lambda}{2} \int a^+ (\hat{u}^+)^2 \\ &\quad - \int a^+ F(\cdot, \tilde{u}) - \int a^+ F(\cdot, \hat{u}^+) \\ &< \liminf I_{\mu_n}(u_{\mu_n}). \end{aligned}$$

We can argue as above to prove that there exists $(\tilde{r}, \tilde{s}, \hat{t}) \in]0, 1]^3 \setminus \{(1, 1, 1)\}$ such that $\check{u} := \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+$ satisfies (\mathcal{N}_i) . The function \check{u} also satisfies (\mathcal{N}_{ii}) . Using first the hypothesis that one of the inequalities (3.12) is strict, then Lemmas 2.5 and 2.9, and finally Lemma 3.1 applied to u_{μ_n} ,

$$\begin{aligned} I_{\mu_n}(\check{u}) &< \liminf I_{\mu_n}(\tilde{r}\tilde{u}_{\mu_n}^+ - \tilde{s}\tilde{u}_{\mu_n}^- + \hat{t}\hat{u}_{\mu_n}^+) \\ &= \liminf I_{\mu_n}(\tilde{r}\tilde{u}_{\mu_n}^+ - \tilde{s}\tilde{u}_{\mu_n}^- + \hat{t}\hat{u}_{\mu_n}^+ - \hat{u}_{\mu_n}^- + \bar{u}_{\mu_n} + \underline{u}_{\mu_n}) \\ &\leq \liminf I_{\mu_n}(u_{\mu_n}) \\ &= \liminf \min I_{\mu_n}|_{\mathcal{N}_{\mu_n}}. \end{aligned} \quad (3.13)$$

The function \tilde{u} also satisfies (\mathcal{N}_{iii}) . Obviously, \tilde{u} satisfies $(\mathcal{N}_{iv}) - (\mathcal{N}_{vii})$. Thus \tilde{u} belongs to \mathcal{N}_{μ_n} . This contradicts (3.13) and proves that

$$u_{\mu_n} \rightarrow u \quad \text{in } H_0^1(\Omega).$$

This proves (1.4). Also, from (3.11),

$$I_0(u) = \lim_{\mu_n} I_{\mu_n}(u_{\mu_n}). \quad (3.14)$$

The proof of Theorem 1.1 will be complete once we prove

Claim 3.5. *Let u be as in (1.4). The function $u|_{\tilde{\omega}}$ is a least energy nodal solution in $H_0^1(\tilde{\omega})$ of (1.3), and the function $u|_{\hat{\omega}}$ is a least energy positive solution in $H_0^1(\hat{\omega})$ of (1.3).*

Proof. Suppose $v \in H_0^1(\Omega)$ is such that $v|_{\Omega^+}$ a solution of (1.3), $v|_{\tilde{\omega}}$ and $v|_{\Omega^-}$ are zero, $v|_{\tilde{\omega}}$ is nodal, $v|_{\hat{\omega}}$ is positive, and either

$$I_0(v|_{\tilde{\omega}}) < I_0(u|_{\tilde{\omega}}) \quad \text{or} \quad I_0(v|_{\hat{\omega}}) < I_0(u|_{\hat{\omega}}).$$

Because Ω , $\tilde{\omega}$ and $\hat{\omega}$ are Lipschitz, \tilde{v} coincides with $v|_{\tilde{\omega}}$ in $\tilde{\omega}$, and \hat{v} coincides with $v|_{\hat{\omega}}$ in $\hat{\omega}$. Without loss of generality, we may also assume

$$I_0(v|_{\tilde{\omega}}) \leq I_0(u|_{\tilde{\omega}}) \quad \text{and} \quad I_0(v|_{\hat{\omega}}) \leq I_0(u|_{\hat{\omega}}).$$

Multiplying both sides of (1.3) by \tilde{v}^+ and integrating, by \tilde{v}^- and integrating, and by \hat{v}^+ and integrating, we find $I_0'(v)(\tilde{v}^+) = I_0'(v)(\tilde{v}^-) = I_0'(v)(\hat{v}^+) = 0$. Note that

$$\left(1 - \frac{2}{\theta}\right)(1 - \Lambda)\|\tilde{v} + \hat{v}^+\|^2 \leq I_0(v) < I_0(u) \leq \left(1 - \frac{2}{\theta}\right)(1 - \Lambda)R^2.$$

The function $v \in \mathcal{N}_\mu$ for all μ . From (3.14) we arrive at the contradiction

$$I_0(v) < I_0(u) = \lim_{\mu_n} I_{\mu_n}(u_{\mu_n}) = \lim \min_{\mathcal{N}_{\mu_n}} I_{\mu_n}.$$

Therefore,

$$I_0(v|_{\tilde{\omega}}) \geq I_0(u|_{\tilde{\omega}}) \quad \text{and} \quad I_0(v|_{\hat{\omega}}) \geq I_0(u|_{\hat{\omega}}).$$

We have proved Claim 3.5. □

The proof of Theorem 1.1 is complete. □

4 Appendix

In this Appendix we give a direct proof of Lemma 2.2 and of Claim 2.8.

Proof of Lemma 2.2. Let $\delta > 0$, ζ, ς be such that

$$\frac{1}{p} = \frac{\zeta}{\vartheta} + \frac{1-\zeta}{2^*}, \quad \frac{1}{p} = \frac{\varsigma}{2} + \frac{1-\varsigma}{2^*},$$

$$\hat{C} = \left(\frac{c_{2^*}^{p(1-\varsigma)}}{2C_T^{p\varsigma} R^{p(1-\varsigma)}} \right)^{2/\varsigma},$$

$$\bar{\delta} = \frac{\hat{C}^{1/p} \delta^{2/(p\varsigma)}}{2R^2} \quad \text{and} \quad \hat{\delta} = \left(\frac{c_{2^*}^{p(1-\varsigma)}}{R^{p(1-\varsigma)}} \frac{\hat{C}^{1/2}}{2^{p/2} C_{\bar{\delta}}^{p/2}} \right)^{\vartheta/(p\varsigma)} \delta^{\vartheta/(p\varsigma)}.$$

The constants C_T and $C_{\bar{\delta}}$ are defined below and ϑ is as in (b). First we derive an estimate for the norm of \underline{u} on $L^p(\Omega^-)$. Consider the set

$$S_1 := \{x \in \Omega^- : \text{dist}(x, \mathbb{R}^N \setminus \Omega^-) < 1/n_1\},$$

where n_1 is large enough so that

$$|S_1| \leq \left(\frac{c_{2^*}^{\vartheta} \hat{\delta}}{3R^{\vartheta}} \right)^{2^*/(2^*-\vartheta)}.$$

Using the Hölder inequality, in the first place we note

$$\int_{S_1} |\underline{u}|^{\vartheta} \leq |\underline{u}|_{2^*}^{\vartheta} |S_1|^{(2^*-\vartheta)/2^*} \leq \frac{\|\underline{u}\|^{\vartheta}}{c_{2^*}^{\vartheta}} |S_1|^{(2^*-\vartheta)/2^*} \leq \frac{\hat{\delta}}{3}. \quad (4.1)$$

Let $\beta > 0$ be a constant such that $a^- \geq \beta$ on $\Omega^- \setminus S_1$. Consider now

$$S_2 := \left\{ x \in \Omega^- : |\underline{u}(x)| \leq \left(\frac{\hat{\delta}}{3|\Omega|} \right)^{1/\vartheta} \right\}.$$

In the second place we note

$$\int_{S_2} |\underline{u}|^{\vartheta} \leq \frac{\hat{\delta}}{3}. \quad (4.2)$$

Let $c_{\hat{\delta}} > 0$ be a constant such that

$$G(x, u) \geq c_{\hat{\delta}} |u|^{\vartheta}, \quad \text{for } x \in \Omega^- \setminus S_1 \text{ and } |u| \geq \left(\frac{\hat{\delta}}{3|\Omega|} \right)^{1/\vartheta}.$$

The existence of such a constant is implied by (b). In the third place we note that

$$\begin{aligned} I_{\mu}(v) + 1 + \frac{\lambda}{2} \int a^+ u^2 + \int a^+ F(\cdot, u) &\geq \mu \int_{\Omega^- \setminus (S_1 \cup S_2)} a^- G(\cdot, u) \\ &\geq \mu \beta c_{\hat{\delta}} \int_{\Omega^- \setminus (S_1 \cup S_2)} |\underline{u}|^{\vartheta}, \end{aligned}$$

and so, for $\mu \geq \mu_\delta := \frac{3(I_\mu(v)+1+C)}{\beta c_\delta \hat{\delta}}$, where C is such that $\frac{\lambda}{2} \int a^+ u^2 + \int a^+ F(\cdot, u) \leq C$,

$$\int_{\Omega^- \setminus (S_1 \cup S_2)} |\underline{u}|^\vartheta \leq \frac{\hat{\delta}}{3}. \quad (4.3)$$

Combining (4.1), (4.2) and (4.3),

$$\int_{\Omega^-} |\underline{u}|^\vartheta \leq \hat{\delta},$$

for $\mu \geq \mu_\delta$. Interpolating the $L^p(\Omega)$ norm between the $L^\vartheta(\Omega)$ and the $L^{2^*}(\Omega)$ norms,

$$\begin{aligned} \int_{\Omega^-} |\underline{u}|^p &\leq \left(\int_{\Omega^-} |\underline{u}|^\vartheta \right)^{p\zeta/\vartheta} |\underline{u}|_{2^*}^{p(1-\zeta)} \\ &\leq \hat{\delta}^{p\zeta/\vartheta} \frac{R^{p(1-\zeta)}}{C_{2^*}^{p(1-\zeta)}} \\ &= \frac{\hat{C}^{1/2}}{2^{p/2} C_{\hat{\delta}}^{p/2}} \delta^{1/\zeta} \leq \frac{\delta}{2} \end{aligned} \quad (4.4)$$

for small δ , and $\mu \geq \mu_\delta$. Now we turn to the estimate for the norm of \underline{u} on $L^p(\Omega^+)$. Let q be the trace exponent $q = 2(N-1)/(N-2)$. If $N = 1$ or 2 we take q to be greater than 2 . There exists $C_{\bar{\delta}}$ such that

$$\left(\int_{\partial\Omega^-} |\underline{u}|^q \right)^{2/q} \leq \bar{\delta} \int_{\Omega^-} |\nabla \underline{u}|^2 + C_{\bar{\delta}} \left(\int_{\Omega^-} |\underline{u}|^p \right)^{2/p}.$$

This follows from [2, bottom of p. 112]. From the expression for $\bar{\delta}$ and (4.4),

$$\left(\int_{\partial\Omega^-} |\underline{u}|^q \right)^{2/q} \leq \hat{C}^{1/p} \delta^{2/(p\varsigma)},$$

for $\mu \geq \mu_\delta$. Using [15, inequality (7.28) on p. 203],

$$\begin{aligned} \left(\int_{\Omega^+} |\underline{u}|^2 \right)^{1/2} &\leq C \|\underline{u}\|_{H^{-1/2}(\partial\Omega^+)} \\ &\leq C \|\underline{u}\|_{L^2(\partial\Omega^+)} \\ &\leq C_T \|\underline{u}\|_{L^q(\partial\Omega^+)} \\ &= C_T \|\underline{u}\|_{L^q(\partial\Omega^-)} \\ &\leq C_T \hat{C}^{1/(2p)} \delta^{1/(p\varsigma)}, \end{aligned}$$

for $\mu \geq \mu_\delta$. This implies

$$\int_{\Omega^+} |\underline{u}|^p \leq \left(\int_{\Omega^+} |\underline{u}|^2 \right)^{p\varsigma/2} |\underline{u}|_{2^*}^{p(1-\varsigma)} \leq C_T^{p\varsigma} \hat{C}^{\varsigma/2} \delta \left(\frac{R}{c_{2^*}} \right)^{p(1-\varsigma)} = \frac{\delta}{2}, \quad (4.5)$$

for $\mu \geq \mu_\delta$. Inequalities (4.4) and (4.5) together finally give

$$|\underline{u}|_p^p \leq \delta,$$

for $\mu \geq \mu_\delta$. □

Proof of Claim 2.8. Consider

$$\varepsilon_1 = \left(\frac{\kappa_2}{4 \sup a^+ |\Omega|} \right)^{1/2}$$

and $\underline{\mathcal{S}}$ as in (2.10). From (2.11),

$$\int_{\{x \in \Omega^+ : |w(x)| \geq \varepsilon_1\} \setminus \underline{\mathcal{S}}} a^+ w^2 \geq \frac{\kappa_2}{2} - \sup a^+ \varepsilon_1^2 |\Omega| = \frac{\kappa_2}{4},$$

for large μ . Let

$$M = \left(\frac{8R^{2^*} \sup a^+}{\kappa_2 c_{2^*}^{2^*}} \right)^{1/(2^*-2)}.$$

Since, by Chebyshev's inequality,

$$|\{x \in \Omega^+ : |w(x)| > M\}| \leq \frac{|w|_{2^*}^{2^*}}{M^{2^*}} \leq \frac{R^{2^*}}{c_{2^*}^{2^*} M^{2^*}},$$

we have

$$\begin{aligned} \int_{\{x \in \Omega^+ : |w(x)| > M\}} a^+ w^2 &\leq \sup a^+ |w|_{2^*}^{2^*} |\{x \in \Omega^+ : |w(x)| > M\}|^{(2^*-2)/2^*} \\ &\leq \sup a^+ \frac{R^{2^*}}{c_{2^*}^{2^*} M^{2^*-2}} \\ &= \frac{\kappa_2}{8}. \end{aligned}$$

Choosing $\underline{\mathcal{S}}$ as in (2.10),

$$\int_{\{x \in \Omega^+ : \varepsilon_1 \leq |w(x)| \leq M\} \setminus \underline{\mathcal{S}}} a^+ w^2 \geq \frac{\kappa_2}{4} - \frac{\kappa_2}{8} = \frac{\kappa_2}{8},$$

for large μ . Let $c'_f > 0$ be such that $f'(x, w) - \frac{f(x, w)}{w} \geq c'_f$ for $|w| \in [\eta \varepsilon_1, M/\eta]$ and $x \in \Omega^+ \setminus \underline{\mathcal{S}}$. Then $\check{f}'(t; w) \geq \eta c'_f \kappa_2 / 8$ for $t \in [\eta, 1/\eta]$ and μ large. This proves Claim 2.8. □

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